Computing 1st Chern Class and Relative π_2 of $\mathbb{C}P^n$

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1 Chern Classes

In general, it's rather difficult to compute the 1st Chern class of a bundle if we don't know anything about how the bundle decomposes. In the case of Grassmanians however, we do know a little more. Consider a k-plane inside $\mathbb{C}^n$. What is the tangent space? The tangent space should consist of "velocity" vectors of how a k-plane P moves. Thus, it should be moving in some orthogonal direction to P. Thus, $T_P Gr_{\mathbb{C}}(k,n) = \mathrm{Hom}(P, P^{\perp})$. The tangent bundle then is $TGr_{\mathbb{C}}(k,n) = \mathrm{Hom}(\xi_k, \xi_k^{\perp})$ where ξ_k is the tautological bundle of k-planes and $\xi_k^{\perp}$ denotes the orthogonal of the tautological bundle. Note this works also for real Grassmanians.

In the case of $\mathbb{C}P^n$, $k = 1$. Now, the 1st Chern class is stable under direct products with a trivial bundle. An example of a rank 1 trivial bundle is $\mathrm{Hom}(\xi, \xi) \cong \mathbb{C}$. Thus,

$$c_1(\mathbb{C}P^n) = c_1(\mathrm{Hom}(\xi, \xi^{\perp})) = c_1(\mathrm{Hom}(\xi, \xi^{\perp}) \oplus \mathrm{Hom}(\xi, \xi)) = c_1(\mathrm{Hom}(\xi, \xi \oplus \xi^{\perp}))$$

Since $\xi \oplus \xi^{\perp} = \mathbb{C}^{n+1}$ (by this, we mean a trivial rank $n+1$ bundle), then

$$c_1(\mathbb{C}P^n) = c_1(\bigoplus^{n+1} \mathrm{Hom}(\xi, \mathbb{C})) = \sum^{n+1} c_1(\xi^*)$$

where $\xi^* = \mathrm{Hom}(\xi, \mathbb{C})$, the dual of the tautological line bundle. So we just need to compute $c_1(\xi^*)$. I believe that $\xi^* = \mathcal{O}(1)$ and $c_1(\xi^*) = +1$ (here, I think we're taking $+1 = h \in \mathbb{Z} \cong H^2(\mathbb{C}P^n, \mathbb{Z})$ to be the Poincaré dual of some hyperplane in $\mathbb{C}P^n$). Thus, $c_1(\mathbb{C}P^n) = (n+1)h$. For other Chern classes, there seems to be a formula. Let $h^i = h \cup ... \cup h$ i times. Then,

$$c_i(\mathbb{C}P^n) = \binom{n+1}{i} h^i.$$

2 Relative Homotopy Groups

When it comes to relative homotopy, we have a long exact sequence. Suppose we're looking at $Y \subset X$ and want to compute $\pi_n(X, Y)$. Then we have an exact sequence

$$... \longrightarrow \pi_n(Y) \xrightarrow{i_*} \pi_n(X) \xrightarrow{\iota_*} \pi_n(X, Y) \xrightarrow{\delta} \pi_{n-1}(Y) \longrightarrow ...$$

i_* is induced from the inclusion map $i : Y \to X$, ι_* is induced also from an inclusion map $\iota : (X, pt) \to (X, Y)$. δ is a boundary map.

Let's try computing $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$ and $\pi_2(\mathbb{C}P^n, T^n)$.

2.1 $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$

The long exact sequence is:

$$... \longrightarrow \pi_2(\mathbb{R}P^n) \longrightarrow \pi_2(\mathbb{C}P^n) \longrightarrow \pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \longrightarrow \pi_1(\mathbb{R}P^n) \longrightarrow ...$$

Recall that the higher homotopy groups of a space are abelian and isomorphic to the higher homotopy groups of its universal cover. When $n \geq 3$, then $\pi_1(S^n) = \pi_2(S^n) = 0$. Also, $\pi_2(\mathbb{C}P^n) = \mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

If the sequence splits; i.e. there is a section $\lambda : \mathbb{Z}^2 \to \pi_2(\mathbb{C}P^n, \mathbb{R}P^n)$, then $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \cong \mathbb{Z} \times \mathbb{Z}_2$. If not, then $\pi_2(\mathbb{C}P^n, \mathbb{R}P^n) \cong \mathbb{Z}$. It's not clear to me which it should be but Jiahao and Alex made some arguments from algebraic topology that it is $\mathbb{Z}$. I'll try to record it as best as I can below:

Claim: $\pi_2(\mathbb{C}P^\infty, \mathbb{R}P^\infty) = \pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$. Reducing from infinity to $n \geq 2$ is an exercise.

$\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$; let $p : K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ be the map which induces on homotopy, the map on $\mathbb{Z} \to \mathbb{Z}$ which is just multiplication by 2. Thinking of this map as a fibration, we can compute the long exact sequence of homotopy groups.

$$\begin{tikzcd} F \arrow[r, hook] & K(\mathbb{Z},2) \arrow[d, "2\cdot"] \\ & K(\mathbb{Z},2) \end{tikzcd}$$

Since $\pi_k K(\mathbb{Z}, 2) = 0$ for $k \neq 2$, then $\pi_k(F) = 0$ for $k \geq 3$. We can argue easily that $\pi_2(F) = 0$ as well. Lastly, we have multiplication by 2 from $\mathbb{Z} \to \mathbb{Z}$ and this is an exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \pi_1(F) \to 0.$$

So in fact, $\pi_1(F) = \mathbb{Z}_2$ and thus, $F = K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$. Claim: this fibrationo inclusion is homotopic to the natural inclusion because they are both induced by the Bockstein of first Stiefel-Whitney class $\beta w_1 \in H^2(\mathbb{R}P^\infty; \mathbb{Z})$ (I don't know what this means).

So it suffices to compute the relative homotopy group of the fiber inclusion, but that's the same as the homotopy group of the base. Here we use

Lemma 2.1. *If $F \to E \to B$ is a fibration, then $\pi_k(E, F) \cong \pi_k(B)$.*

Proof. The proof of this lemma is looking at long exact sequence of pairs and long exact sequence of fibrations, then applying the five lemma.

$$\begin{tikzcd} ...\pi_k(F) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_k(E) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_k(E,F) \arrow[r] \arrow[d, "\cong"] & \pi_{k-1}(F) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_{k-1}(E)... \arrow[d, "\mathrm{id}"] \\ ...\pi_k(F) \arrow[r] & \pi_k(E) \arrow[r] & \pi_k(B) \arrow[r] & \pi_{k-1}(F) \arrow[r] & \pi_{k-1}(E)... \end{tikzcd}$$

Note there is a natural map $\pi_k(E, F) \to \pi_k(B)$ via projection $(E, F) \to (B, point)$. □

Thus, $\pi_2(\mathbb{C}P^\infty, \mathbb{R}P^\infty) \cong \pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$.

2.2 $\pi_2(\mathbb{C}P^n, T^n)$

Let's consider any $n \geq 1$. The universal cover of T^n is $\mathbb{R}^n$ so $\pi_2(T^n) = 0$ and $\pi_1(T^n) = \mathbb{Z}^n$.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(\mathbb{C}P^n, T^n) \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

gives a surjection $\pi_2(\mathbb{C}P^n, T^n) \to \mathbb{Z}^n$.